Journal of Computational and Applied Mathematics 17 (1987) 359–364
North-Holland

A fourth-order finite difference method based on uniform mesh for singular two-point boundary-value problems

M.M. CHAWLA
Department of Mathematics, Indian Institute of Technology, Hauz Khas, New Delhi-110016, India

Received 27 January 1986
Revised 7 August 1986

Abstract: We present a new fourth-order finite difference method based on uniform mesh for the (weakly) singular two-point boundary value problem: $(x^\alpha y')' = f(x, y)$, $0 < x \leqslant 1$, $y(0) = A$, $y(1) = B$, $0 < \alpha < 1$. Our method provides $O(h^4)$-convergent approximations for *all* $\alpha \in (0, 1)$; for $\alpha = 0$ it reduces to the well-known fourth-order method of Numerov for $y'' = f(x, y)$.

Keywords: Singular two-point boundary value problems, fourth-order finite difference method, uniform mesh, Numerov method.

1. Introduction

Consider the (weakly) singular two-point boundary-value problem:

$$\begin{aligned} &(x^\alpha y')' = f(x, y), \quad 0 < x \leqslant 1, \\ &y(0) = A, \qquad y(1) = B. \end{aligned} \tag{1}$$

Here, $\alpha \in (0, 1)$ and A, B are finite constants. We assume that, for $(x, y) \in \{[0, 1] \times \mathbb{R}\}$, $f(x, y)$ is continuous, $\partial f/\partial y$ exists and is continuous and $\partial f/\partial y \geqslant 0$. Recently, Chawla and Katti [1] described a new finite difference method for the singular two-point boundary-value problem (1) which was based on uniform mesh and, under quite general conditions on f, was shown to provide $O(h^2)$-convergent approximations for all $\alpha \in (0, 1)$. This method was based on one evaluation of f and for $\alpha = 0$ it reduces to the classical second-order method for $y'' = f(x, y)$.

Singular boundary-value problems have been considered by various authors; some of these works are mentioned in the introductions to our earlier papers [1] and [2]. In addition, we note the interesting work of De Hoog and Weiss [3] who examined the application of collocation methods based on piecewise polynomials to the numerical solution of boundary-value problems for systems of ordinary differential equations with singularity of the first kind. The schemes are shown to be stable and convergent. Enhanced accuracy at the nodes for suitably chosen collocation points (superconvergence) is established for a class of important problems. This paper gives also higher-order methods.

0377-0427/87/$3.50

The purpose of this paper is to give a new fourth-order finite difference method based on uniform mesh for the (weakly) singular two-point boundary value problem (1). The present method is based on three evaluations of f. Our method is shown to provide $O(h^4)$-convergent approximations for *all* $\alpha \in (0, 1)$; for $\alpha = 0$ it reduces to the well-known fourth-order method of Numerov for $y'' = f(x, y)$. The method and its fourth-order convergence is illustrated by an example.

2. The finite difference method

For a positive integer $N \geqslant 2$, consider the uniform mesh over $[0, 1]: x_k = kh$, $k = 0(1)N$, $h = 1/N$. Let $y_k = y(x_k)$, $f_k = f(x_k, y_k)$, etc. We start with the following identity obtained in Chawla and Katti [2]:

$$\frac{y_{k+1} - y_k}{J_k} - \frac{y_k - y_{k-1}}{J_{k-1}} = \frac{I_k^+}{J_k} + \frac{I_k^-}{J_{k-1}}, \quad k = 1(1)N-1, \tag{2}$$

where

$$\begin{aligned} I_k^+ &= \frac{1}{1-\alpha} \int_{x_k}^{x_{k+1}} \left(x_{k+1}^{1-\alpha} - t^{1-\alpha}\right) f(t)\, \mathrm{d}t, \\ I_k^- &= \frac{1}{1-\alpha} \int_{x_{k-1}}^{x_k} \left(t^{1-\alpha} - x_{k-1}^{1-\alpha}\right) f(t)\, \mathrm{d}t, \end{aligned} \tag{3}$$

and

$$J_k = \left(x_{k+1}^{1-\alpha} - x_k^{1-\alpha}\right)/(1-\alpha). \tag{4}$$

Now, by Taylor expansion of f about x_k, we obtain

$$I_k^{\pm} = \sum_{i=0}^{3} \frac{1}{i!} A_{i,k}^{\pm} f_k^{(i)} + \frac{1}{4!} A_{4,k}^{\pm} f^{(4)}(\xi_k^{\pm}), \quad \xi_k^{\pm} \in (x_k, x_{k\pm1}), \tag{5}$$

and

$$\begin{aligned} A_{i,k}^+ &= \frac{1}{1-\alpha} \int_{x_k}^{x_{k+1}} \left(x_{k+1}^{1-\alpha} - t^{1-\alpha}\right)(t - x_k)^i\, \mathrm{d}t, \quad i = 0(1)4, \\ A_{i,k}^- &= \frac{1}{1-\alpha} \int_{x_{k-1}}^{x_k} \left(t^{1-\alpha} - x_{k-1}^{1-\alpha}\right)(t - x_k)^i\, \mathrm{d}t, \quad i = 0(1)4. \end{aligned} \tag{6}$$

Note that $A_{m,k}^{\pm} > 0$, $m = 0, 2, 4$, $A_{mk}^+ > 0$, $m = 1, 3$, $A_{m,k}^- < 0$, $m = 1, 3$, and $J_k > 0$. With the help of (5) from (2) we obtain

$$\frac{y_{k+1} - y_k}{J_k} - \frac{y_k - y_{k-1}}{J_{k-1}} = \sum_{i=0}^{3} \frac{1}{i!} B_{i,k} f_k^{(i)} + \frac{1}{4!} B_{4,k} f^{(4)}(\xi_k), \quad k = 1(1)N-1, \tag{7}$$

where $\xi_k \in (x_{k-1}, x_{k+1})$ and

$$B_{i,k} = A_{i,k}^+/J_k + A_{i,k}^-/J_{k-1}, \quad i = 0(1)4. \tag{8}$$

Note that $B_{m,k} > 0$, $m = 0, 2, 4$.

Now, in (7) replacing f_k' and f_k'' by their central difference approximations given by

$$f_k' = \frac{f_{k+1} - f_{k-1}}{2h} - \tfrac{1}{6}h^2 f^{(3)}(\eta_k), \tag{9}$$

and

$$f_k'' = \frac{f_{k+1} - 2f_k + f_{k-1}}{h^2} - \tfrac{1}{12}h^2 f^{(4)}(\zeta_k), \tag{10}$$

where η_k, $\zeta_k \in (x_{k-1}, x_{k+1})$, we obtain

$$\begin{aligned} &-\frac{1}{J_{k-1}}y_{k-1} + \left(\frac{1}{J_k} + \frac{1}{J_{k-1}}\right)y_k - \frac{1}{J_k}y_{k+1} + \frac{1}{2h}\left(-B_{1,k} + \frac{1}{h}B_{2,k}\right)f_{k-1} \\ &\quad + \left(B_{0,k} - \frac{1}{h^2}B_{2,k}\right)f_k + \frac{1}{2h}\left(B_{1,k} + \frac{1}{h}B_{2,k}\right)f_{k+1} + t_k(h) = 0, \quad k = 1(1)N-1, \end{aligned} \tag{11}$$

where

$$t_k(h) = -\tfrac{1}{6}h^2 B_{1,k} f^{(3)}(\eta_k) + \tfrac{1}{6}B_{3,k} f_k^{(3)} - \tfrac{1}{24}h^2 B_{2,k} f^{(4)}(\zeta_k) + \tfrac{1}{24}B_{4,k} f^{(4)}(\xi_k). \tag{12}$$

We note that

$$A_{i,k}^{\pm} = \frac{1}{i+1}\sum_{j=0}^{i+1}\frac{(-1)^j}{i+2-\alpha-j}\binom{i+1}{j}x_k^j\left(x_{k\pm1}^{i+2-\alpha-j} - x_k^{i+2-\alpha-j}\right), \quad i = 0, 1, 2, \tag{13}$$

and the $B_{i,k}$, $i = 0, 1, 2$, required in the discretizations (11) can be obtained from (8) with the help of (13) and (4).

The discretizations given by (11) are the required discretizations of the singular two-point boundary-value problem (1). Note that each discretization is based on three function evaluations. Since for $\alpha = 0$, $B_{0,k} = h$, $B_{1,k} = 0$ and $B_{2,k} = \frac{1}{6}h^3$, therefore the method based on the discretizations (11) reduces to the well-known Numerov's method for $y'' = f(x, y)$. In the next section we show that the present method based on the discretizations (11) does provide $O(h^4)$-convergent approximations for all $\alpha \in (0, 1)$.

3. $O(h^4)$-convergence of the method for all $\alpha \in (0, 1)$

Let $D = (d_{ij})$ denote the tridiagonal matrix with

$$d_{k,k-1} = -\frac{1}{J_{k-1}}, \quad k = 2(1)N-1, \qquad d_{k,k} = \frac{1}{J_k} + \frac{1}{J_{k-1}}, \quad k = 1(1)N-1,$$

$$d_{k,k+1} = -\frac{1}{J_k}, \quad k = 1(1)N-2.$$

Let $P = (p_{ij})$ denote the tridiagonal matrix with

$$p_{k,k-1} = \frac{1}{2h}\left(-B_{1,k} + \frac{1}{h}B_{2,k}\right), \quad k = 2(1)N-1, \qquad p_{k,k} = B_{0,k} - \frac{1}{h^2}B_{2,k},$$

$k = 1(1)N-1$,

$$p_{k,k+1} = \frac{1}{2h}\left(B_{1,k} + \frac{1}{h}B_{2,k}\right), \quad k = 1(1)N-2.$$

Also, let $Y=(y_1,\dots,y_{N-1})^{\mathrm{T}}$, $F(Y)=(f_1,\dots,f_{N-1})^{\mathrm{T}}$, $T(h)=(t_1(h),\dots,t_{N-1}(h))^{\mathrm{T}}$ and let $Q=(q_1,\dots,q_{N-1})^{\mathrm{T}}$ where

$$q_1=\frac{A}{J_0}-\frac{1}{2h}\left(-B_{1,1}+\frac{1}{h}B_{2,1}\right)f_0,$$

$$q_{N-1}=\frac{B}{J_{N-1}}-\frac{1}{2h}\left(B_{1,N-1}+\frac{1}{h}B_{2,N-1}\right)f_N,\qquad q_i=0,\quad i=2(1)N-2.$$

Then the discretizations (11) can be expressed in matrix form as:

$$DY+PF(Y)+T(h)=Q. \tag{14}$$

Thus a method based on (14) provides approximation $\tilde{Y}$ for Y by solving the $(N-1)\times(N-1)$ system:

$$D\tilde{Y}+PF(\tilde{Y})=Q. \tag{15}$$

Now, let $E=\tilde{Y}-Y$. We may write $F(\tilde{Y})-F(Y)=UE$ where $U=\operatorname{diag}\{u_1,\dots,u_{N-1}\}$; note that $u_k\geqslant 0$. Then, from (14) and (15) we obtain the error equation:

$$(D+PU)E=T(h). \tag{16}$$

From (6) and (4) it can be shown that for fixed x_k and $h\to 0$,

$$\begin{aligned}A^{+}_{i,k}&=\frac{1}{i+1}h^{i+2}x_k^{-\alpha}\left[\frac{1}{i+2}-\frac{1}{i+3}\frac{\alpha h}{x_k}+\mathrm{O}(h^2)\right],\\ A^{-}_{i,k}&=(-1)^i\frac{1}{i+1}h^{i+2}x_k^{-\alpha}\left[\frac{1}{i+2}+\frac{1}{i+3}\frac{\alpha h}{x_k}+\mathrm{O}(h^2)\right],\end{aligned} \tag{17}$$

and

$$\begin{aligned}J_k&=hx_k^{-\alpha}\left[1-\tfrac{1}{2}\alpha\frac{h}{x_k}+\tfrac{1}{6}\alpha(\alpha+1)h^2/x_k^2+\mathrm{O}(h^3)\right],\\ J_{k-1}&=hx_k^{-\alpha}\left[1+\tfrac{1}{2}\alpha\frac{h}{x_k}+\tfrac{1}{6}\alpha(\alpha+1)h^2/x_k^2+\mathrm{O}(h^3)\right],\end{aligned} \tag{18}$$

and hence,

$$B_{i,k}=\begin{cases}\dfrac{2}{(i+1)(i+2)}h^{i+1}+\mathrm{O}(h^{i+3}), & i=0,2,4,\\ -\dfrac{\alpha}{(i+2)(i+3)x_k}h^{i+2}+\mathrm{O}(h^{i+4}), & i=1,3.\end{cases} \tag{19}$$

With the help of (19) we obtain

$$\begin{aligned}p_{k,k-1}&=\tfrac{1}{12}h(1+\alpha h/2x_k)+\mathrm{O}(h^3),\qquad p_{k,k}=\tfrac{5}{6}h+\mathrm{O}(h^3),\\ p_{k,k+1}&=\tfrac{1}{12}h(1-\alpha h/2x_k)+\mathrm{O}(h^3).\end{aligned} \tag{20}$$

Also, with the help of (18) we obtain

$$\begin{aligned}d_{k,k-1}&=-(1/h)x_k^{\alpha}\left[1-\alpha h/2x_k+\mathrm{O}(h^2)\right],\qquad d_{k,k}=(1/h)x_k^{\alpha}\left[2+\mathrm{O}(h^2)\right],\\ d_{k,k+1}&=-(1/h)x_k^{\alpha}\left[1+\alpha h/2x_k+\mathrm{O}(h^2)\right].\end{aligned} \tag{21}$$

It is easy to see that the matrices D and $D+PU$ are both irreducible and monotone for sufficiently small h. Now, since $D+PU \geqslant D$, it follows that $(D+PU)^{-1} \leqslant D^{-1}$. From (16) it therefore follows that

$$\| E \| \leqslant \| D^{-1}T(h) \|, \tag{22}$$

in the uniform norm. The inverse $D^{-1}=(d_{ij}^{-1})$ of the matrix D is given (see [1]) by

$$d_{n,m}^{-1}=\begin{cases} x_n^{1-\alpha}(1-x_m^{1-\alpha})/(1-\alpha), & n \leqslant m, \\ x_m^{1-\alpha}(1-x_n^{1-\alpha})/(1-\alpha), & n \geqslant m. \end{cases} \tag{23}$$

Let β be such that $\alpha+\beta<1$. We assume that

$$\max_{0<x\leqslant 1} x^{\beta} | f^{(3)} | \leqslant N_3, \qquad \max_{0<x\leqslant 1} x^{1+\beta} | f^{(4)} | \leqslant N_4, \tag{24}$$

for suitable positive constants N_3 and N_4. From (19) it follows that for fixed x_k and h sufficiently small,

$$B_{i,k} < \frac{4}{(i+1)(i+2)} h^{i+1}, \quad i=1, 3, \qquad | B_{i,k} | < \frac{2\alpha}{(i+2)(i+3)x_k} h^{i+2}, \quad i=1, 3. \tag{25}$$

With the help of (24) and (25) from (12) it now follows that for $k=1(1)N-1$,

$$| t_k(h) | \leqslant Ch^5 x_k^{-1-\beta}, \quad k=1(1)N-1, \tag{26}$$

where

$$C=\tfrac{7}{360}(2\alpha N_3+N_4).$$

Now, following arguments precisely as given in [1, p. 563] we can show that

$$\| E \| \leqslant C^* h^4, \tag{27}$$

where now

$$C^* = C \max\left(\frac{1}{\beta(1-\alpha-\beta)}, \frac{1}{(1-\alpha)^2 e} \right).$$

We have thus shown that our present method based on uniform mesh described by (15) provides $O(h^4)$-convergent approximations for *all* $\alpha \in (0, 1)$.

4. Numerical illustration

To illustrate the present method and its fourth-order rate of convergence, we consider the following two-point boundary-value problem:

$$\begin{aligned} &(x^{\alpha}y')' = sx^{\alpha+s-2}(sx^s \, e^y - (\alpha+s-1))/(4+x^s), \quad 0<x\leqslant 1, \\ &y(0)=\ln \tfrac{1}{4}, \qquad y(1)=\ln \tfrac{1}{5}, \end{aligned} \tag{28}$$

with the exact solution $y(x)=\ln(1/(4+x^s))$.

For this example, $f=O(x^{\alpha+s-2})$, $x \to 0$, and

$$x^{\beta} f^{(3)}, \qquad x^{1+\beta} f^{(4)} = O(x^{\alpha+\beta+s-5}).$$

Table 1

N	$\alpha = 0.25$	0.5	0.75	0.95
64	8.74 (−6)	1.40 (−5)	4.65 (−5)	5.91 (−5)
128	5.48 (−7)	8.83 (−7)	3.06 (−6)	4.22 (−6)
256	3.43 (−8)	5.56 (−8)	1.97 (−7)	2.87 (−7)
512	2.14 (−9)	3.49 (−9)	1.25 (−8)	1.88 (−8)
1024	1.33 (−10)	2.18 (−10)	7.85 (−10)	1.21 (−9)

So, conditions for the fourth-order convergence of the method will be satisfied if

$$\alpha + \beta > 5 - s \quad \text{and} \quad \alpha + \beta < 1.$$

We computed the solution of the problem (28) by our present method (15) for the choice $s = 5$. In Table 1 we show the error-norm computed for (28) for values of $\alpha = 0.25, 0.5, 0.75$ and 0.95, and for a few values of N. The computed results do verify the fourth-order rate of convergence of our present method.

References

[1] M.M. Chawla and C.P. Katti, A uniform mesh finite difference method for a class of singular two-point boundary value problems, *SIAM J. Numer. Anal.* **22** (1985) 561–565.

[2] M.M. Chawla and C.P. Katti, Finite difference methods and their convergence for a class of singular two point boundary value problems, *Numer. Math.* **39** (1982) 341–350.

[3] F.R. de Hoog and R. Weiss, Collocation methods for singular boundary value problems, *SIAM J. Numer. Anal.* **15** (1978) 198–217.